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# Cook, Kannan and Schrijver's example revisited

Yanjun Li [a,*], Jean-Philippe P. Richard [b]

[a] *Krannert School of Management, Purdue University, West Lafayette, IN 47907, USA*

[b] *School of Industrial Engineering, Purdue University, West Lafayette, IN 47907, USA*

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ABSTRACT

In 1990, Cook, Kannan and Schrijver [W. Cook, R. Kannan, A. Schrijver, Chvátal closures for mixed integer programming problems, Mathematical Programming 47 (1990) 155–174] proved that the split closure (the 1st 1-branch split closure) of a polyhedron is again a polyhedron. They also gave an example of a mixed-integer polytope in $\mathbb{R}^{2+1}$ whose 1-branch split rank is infinite. We generalize this example to a family of high-dimensional polytopes and present a closed-form description of the $k$th 1-branch split closure of these polytopes for any $k \geq 1$. Despite the fact that the $m$-branch split rank of the $(m+1)$-dimensional polytope in this family is 1, we show that the 2-branch split rank of the $(m+1)$-dimensional polytope is infinite when $m \geq 3$. We conjecture that the $t$-branch split rank of the $(m+1)$-dimensional polytope of the family is infinite for any $1 \leq t \leq m-1$ and $m \geq 2$.



## 1. Introduction

Disjunctive programming and disjunctive cuts were first introduced by Balas in the 1970s [3,4]. Disjunctive programming is the problem of minimizing a linear function $cx$, where $c, x \in \mathbb{R}^n$, subject to the constraints:

$$Ax \geq b, \tag{1}$$

$$x \geq 0, \tag{2}$$

$$\bigvee_{q \in Q} (D^q x \geq d^q), \tag{3}$$

where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $D^q \in \mathbb{R}^{m_q \times n}$, $d^q \in \mathbb{R}^{m_q}$, $q \in Q$, $Q$ is a index set, and $\bigvee_{q \in Q}(D^q x \geq d^q)$ requires that $x$ satisfies the constraints of at least one of the systems $D^q x \geq d^q$ for $q \in Q$. A variety of integer and nonconvex programs can be written in this form. A disjunctive cut is a valid inequality for the disjunctive programming problem, *i.e.*, an inequality that is satisfied by every $x$ that satisfies (1)–(3).

In this paper, we consider the polyhedron $P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^p \mid Ax + By \leq d\}$, where $A \in \mathbb{Q}^{m \times n}$, $B \in \mathbb{Q}^{m \times p}$ and $d \in \mathbb{Q}^m$. Throughout this paper, $P$ is the LP relaxation of the mixed integer set $P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$. We also consider the following particular type of disjunctions:

$$\bigvee_{S \subseteq \{1,2,\ldots,t\}} (\pi^i x \leq \pi_0^i, \text{ if } i \in S;\ \pi^i x \geq \pi_0^i + 1, \text{ if } i \notin S),$$

where $\pi^i \in \mathbb{Z}^n$, $\pi_0^i \in \mathbb{Z}$, $1 \leq i \leq t$ and $t$ is a positive integer. Because we use $t$ integral vectors $\pi^i$ and $t$ integers $\pi_0^i$ and because each individual disjunction is of the form $\pi^i x \leq \pi_0^i$ and $\pi^i x \geq \pi_0^i + 1$, we define this disjunction as a *t-branch split*

* Corresponding author. Tel.: +1 765 494 4525; fax: +1 765 494 9658.
*E-mail addresses:* li14@purdue.edu (Y. Li), jprichar@ecn.purdue.edu (J.-P.P. Richard).

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*disjunction*. We define a disjunctive cut that is derived from applying a $t$-branch split disjunction to $P$ as a *$t$-branch split cut*. It is easy to see that every $t$-branch split cut of $P$ is a valid inequality for the mixed-integer set $P \cap (\mathbb{Z}^n \times \mathbb{R}^p)$.

Given $t$ vectors $(\pi^i, \pi_0^i) \in \mathbb{Z}^{n+1}$ for $i = 1, \ldots, t$ and given $S \subseteq \{1, \ldots, t\}$, we define $P(S, \pi, \pi_0) = \{(x, y) \in P \mid \pi^i x \leq \pi_0^i \text{ for } i \in S; \pi^i x \geq \pi_0^i + 1 \text{ for } i \notin S\}$, where $\pi = \{\pi^1, \pi^2, \ldots, \pi^t\}$ and $\pi_0 = \{\pi_0^1, \pi_0^2, \ldots, \pi_0^t\}$. We then define the 1st *$t$-branch split closure* of $P$ as:

$$P^{(1,t)} = \bigcap_{(\pi^i,\pi_0^i)\in\mathbb{Z}^{n+1} \quad \text{for } i=1,\ldots,t} \operatorname{conv}\left(\bigcup_{S\subseteq\{1,\ldots,t\}} P(S, \pi, \pi_0)\right).$$

Given an integer $k \geq 2$, we define the $k$th $t$-branch split closure $P^{(k,t)}$ of $P$ iteratively as $P^{(k,t)} = (P^{(k-1,t)})^{(1,t)}$. We denote the integral hull of $P$ by $P_I$. Finally, we define the $t$-branch split rank of $P$ as the smallest integer $l$ such that $P^{(l,t)} = P_I$, if such an integer exists. Otherwise, $P^{(k,t)}$ contains $P_I$ as a proper subset for all $k \geq 0$ and we say that the $t$-branch split rank of $P$ is infinite. In our notation, the classical split disjunction introduced by Cook, Kannan and Schrijver [5] is referred to as 1-branch split disjunction, while the traditional split cut is referred to as 1-branch split cut. Further, the *split closure* of a polyhedron is referred to as its 1st 1-branch split closure in our nomenclature.

In their paper, Cook, Kannan and Schrijver proved that the 1st 1-branch split closure of $P$ is again a polyhedron. They also described a special polytope $P$ in $\mathbb{R}^{2+1}$, whose 1-branch split rank is infinite. This polytope $P$ is the convex hull of the four vectors $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$ and $(\frac{1}{2}, \frac{1}{2}, \epsilon)$, where $\epsilon$ is a rational that satisfies $0 < \epsilon < 1$, the first two variables are integers and the third variable is continuous. They gave a sketch of a proof that there exists $\epsilon_1$ satisfying $0 < \epsilon_1 < \epsilon$ for which $(\frac{1}{2}, \frac{1}{2}, \epsilon_1) \in P^{(1,1)}$. Hence, $P^{(1,1)}$ contains a polytope of the same form as $P$. Applying the argument recursively, $P^{(k,1)}$ must contain a vector $(\frac{1}{2}, \frac{1}{2}, \epsilon_k)$ for some $0 < \epsilon_k < 1$ and for any integer $k \geq 2$. Note that because $P_I$ is simply the convex hull of the vectors $(0, 0, 0)$, $(2, 0, 0)$ and $(0, 2, 0)$, $P^{(k,1)} \supset P_I$ for any finite $k$. Therefore, the 1-branch split rank of $P$ is infinite.

In this paper, we generalize the polytope given by Cook, Kannan and Schrijver to a family of high-dimensional polytopes, and derive a closed-form description of the $k$th 1-branch split closure of each polytope in this family. This result is interesting as it gives an easy way to compare the strength of mixed-integer cutting planes on a family of problems that have infinite split cut rank even if inequalities obtained through a finite number of rounds of split disjunctive arguments are added.

In contrast, we show that the $m$-branch split rank of the $(m + 1)$-dimensional polytope in this family is 1, which implies that applying multi-branch split cuts can provide significant gains in solving mixed-integer programming problems. However, we also prove that the 2-branch split rank of the $(m+1)$-dimensional polytope in the family is infinite, whenever $m \geq 3$. This extends Cook, Kannan and Schrijver's result about infinite rank of 1-branch split cut to multi-branch split cuts. We conjecture that the $t$-branch split rank of the $(m+1)$-dimensional polytope in this family is infinite for any $1 \leq t \leq m-1$ and $m \geq 2$.

The remainder of the paper is organized as follows. In Section 2, we generalize the polytope given by Cook, Kannan and Schrijver to a family of high-dimensional polytopes and derive a closed-form expression for their $k$th 1-branch split closure. In Section 3, we prove two results about finite and infinite multi-branch split ranks and propose two related conjectures. We also discuss how to extend the above results to a family of unbounded polyhedra. Finally, in Section 4, we show how to obtain the inequalities of the integer hull $P_I$ using a "nonlinear" 1-branch split disjunction. We also mention some related results on the $\infty$-convergent convexification procedure given by Owen and Mehrotra [8], and the set of mixed-integer feasible solutions for two rows of a simplex tableau studied by Andersen, Louveaux, Weismantel and Wolsey [2].

## 2. The $k$th 1-branch split closure

In this section, we first generalize Cook, Kannan and Schrijver's polytope to a family of polytopes in high dimension. Then we derive a closed-form expression for the $k$th 1-branch split closure of these polytopes.

For $m \geq 2$, the $(m + 1)$-dimensional polytope of the family is defined as the convex hull of the following $m + 2$ vectors $\mathbb{R}^{m+1}$: $(0, 0, \ldots, 0)$, $(m, 0, \ldots, 0)$, $(0, m, 0, \ldots, 0), \ldots, (0, 0, \ldots, m, 0)$ and $(\frac{m-1}{m}, \ldots, \frac{m-1}{m}, \frac{m-1}{m}, \epsilon)$ for some rational $0 < \epsilon < 1$. We denote this polytope by $P(m+1, \frac{m-1}{m}, \epsilon)$. The first $m$ variables are integer and the last variable is continuous, so the corresponding mixed-integer set is $P(m + 1, \frac{m-1}{m}, \epsilon) \cap (\mathbb{Z}^m \times \mathbb{R})$. The explicit formulation of $P(m + 1, \frac{m-1}{m}, \epsilon)$ using inequalities is:

$$P\left(m + 1, \frac{m-1}{m}, \epsilon\right) = \left\{(x, y) \in \mathbb{R}^m \times \mathbb{R}_+ \;\middle|\; \begin{array}{l} \displaystyle\sum_{j=1}^{m} \epsilon x_j + y \leq m\epsilon, \\ \epsilon x_j - \dfrac{m-1}{m} y \geq 0, \forall j = 1, \ldots, m \end{array}\right\}.$$

It is easily verified that $P(3, \frac{1}{2}, \epsilon)$ is the example studied by Cook, Kannan and Schrijver.

In order to simplify the derivation of the $k$th 1-branch split closure of $P(m + 1, \frac{m-1}{m}, \epsilon)$ for $m \geq 2$, we consider a more general polytope $P(m + 1, \alpha, \epsilon)$ in $\mathbb{R}^{m+1}$, defined as the convex hull of the $m + 2$ vectors: $\mathcal{V}_0 \equiv (0, 0, \ldots, 0)$,

$\mathcal{V}_1 \equiv (m, 0, \dots, 0)$, $\mathcal{V}_2 \equiv (0, m, 0, \dots, 0), \dots, \mathcal{V}_m \equiv (0, 0, \dots, m, 0)$ and $\mathcal{V}_{m+1} \equiv (\alpha, \dots, \alpha, \alpha, \epsilon)$, where $0 < \epsilon < 1$ is a rational and $\frac{m-1}{m} \le \alpha < 1$. The inequality description of $P(m+1, \alpha, \epsilon)$ is given by:

$$P(m+1, \alpha, \epsilon) = \left\{ (x, y) \in \mathbb{R}^m \times \mathbb{R}_+ \left| \begin{array}{l} \sum_{j=1}^{m} \epsilon x_j + m(1-\alpha)y \le m\epsilon, \\ \epsilon x_j - \alpha y \ge 0, \forall j = 1, \dots, m \end{array} \right. \right\}.$$

In the following sections, we often use the following two simple observations that we record for easy reference. The first one simply states that the integer points of $P(m+1, \alpha, \epsilon)$ are not cut off by split cuts.

**Observation 1.** *Assume that* $(x_1, \dots, x_m, y) \in P(m+1, \alpha, \epsilon) \cap (\mathbb{Z}^m \times \mathbb{R})$. *Then* $(x_1, \dots, x_m, y) \in P^{(k,t)}(m+1, \alpha, \epsilon)$ *for all* $k$ *and* $t$.

In particular, since $\mathcal{V}_0, \mathcal{V}_1, \dots, \mathcal{V}_m$ belong to the integral hull of $P(m+1, \alpha, \epsilon)$, they also belong to its $k$th $t$-branch split closure for all $k$ and for all $t$. The following observation then follows.

**Observation 2.** *Assume that* $(x_1, \dots, x_m, y) \in P^{(k,t)}(m+1, \alpha, \epsilon)$ *for some* $k$ *and* $t$. *Then* $(x_1, \dots, x_m, y') \in P^{(k,t)}(m+1, \alpha, \epsilon)$ *for all* $y' \in [0, y)$.

Next we give a closed-form formulation for the 1st 1-branch split closure of $P(m+1, \alpha, \epsilon)$.

**Proposition 1.** $P^{(1,1)}(m+1, \alpha, \epsilon) = P(m+1, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon)$.

**Proof.** $\supseteq$: First, we show that $P^{(1,1)}(m+1, \alpha, \epsilon) \supseteq P(m+1, 1-\frac{\alpha}{m}, \frac{m-1}{m}\epsilon)$. To this end, we prove that $(1 - \frac{\alpha}{m}, \dots, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon) \in P^{(1,1)}(m+1, \alpha, \epsilon)$. Using the assumption that $\frac{m-1}{m} \le \alpha < 1$, it can be verified that the following vectors

$$\begin{aligned}
v_1 &= \left(1, \frac{(m-1)\alpha}{m-\alpha}, \dots, \frac{(m-1)\alpha}{m-\alpha}, \frac{(m-1)\epsilon}{m-\alpha}\right), \\
v_2 &= \left(\frac{(m-1)\alpha}{m-\alpha}, 1, \frac{(m-1)\alpha}{m-\alpha}, \dots, \frac{(m-1)\alpha}{m-\alpha}, \frac{(m-1)\epsilon}{m-\alpha}\right), \\
&\dots\dots, \\
v_m &= \left(\frac{(m-1)\alpha}{m-\alpha}, \dots, \frac{(m-1)\alpha}{m-\alpha}, 1, \frac{(m-1)\epsilon}{m-\alpha}\right), \\
w_1 &= \left(0, \frac{m(1-\alpha)}{\alpha}, \dots, \frac{m(1-\alpha)}{\alpha}, 0\right), \\
w_2 &= \left(\frac{m(1-\alpha)}{\alpha}, 0, \frac{m(1-\alpha)}{\alpha}, \dots, \frac{m(1-\alpha)}{\alpha}, 0\right), \\
&\dots\dots, \\
w_m &= \left(\frac{m(1-\alpha)}{\alpha}, \dots, \frac{m(1-\alpha)}{\alpha}, 0, 0\right)
\end{aligned}$$

belong to $P(m+1, \alpha, \epsilon)$ for $1 \le i \le m$. Further, we observe that

$$\left(1 - \frac{\alpha}{m}, \dots, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon\right) = \left(1 - \frac{\alpha}{m}\right) v_i + \frac{\alpha}{m} w_i \tag{4}$$

for $1 \le i \le m$. It is also clear that $w_i$ belongs to $P^{(1,1)}(m+1, \alpha, \epsilon)$ for $1 \le i \le m$ because it can be obtained using a convex combination of the integer points $\mathcal{V}_j$ for $j \in \{0, \dots, m\} \setminus \{i\}$ since $\frac{m-1}{m} \le \alpha < 1$.

Now we define two families of vectors in $\mathbb{R}^m$ that will be used in the remainder of the proof. We denote the first family as

$$\begin{aligned}
\tilde{v}_1 &= \left(1, \frac{(m-1)\alpha}{m-\alpha}, \dots, \frac{(m-1)\alpha}{m-\alpha}\right), \\
\tilde{v}_2 &= \left(\frac{(m-1)\alpha}{m-\alpha}, 1, \frac{(m-1)\alpha}{m-\alpha}, \dots, \frac{(m-1)\alpha}{m-\alpha}\right), \\
&\dots\dots, \\
\tilde{v}_m &= \left(\frac{(m-1)\alpha}{m-\alpha}, \dots, \frac{(m-1)\alpha}{m-\alpha}, 1\right),
\end{aligned}$$

while we denote the second family as

$$\begin{aligned}
&e = (1, 1, 1, \ldots, 1)\,,\\
&\tilde{e}_1 = (1, 0, 0, \ldots, 0)\,,\\
&\tilde{e}_2 = (0, 1, 0, \ldots, 0)\,,\\
&\ldots\ldots,\\
&\tilde{e}_m = (0, 0, \ldots, 0, 1)\,.
\end{aligned}$$

Consider an arbitrary 1-branch split disjunction $(\pi x \le \pi_0) \vee (\pi x \ge \pi_0 + 1)$. Without loss of generality, we assume that $e$ satisfies $\pi x \le \pi_0$. We note that

$$\tilde{v}_i = \frac{(m-1)\alpha}{m-\alpha} e + \left(1 - \frac{(m-1)\alpha}{m-\alpha}\right) \tilde{e}_i \tag{5}$$

for $1 \le i \le m$, where $0 < \frac{(m-1)\alpha}{m-\alpha} < 1$.

First, we assume that $\tilde{e}_i$ satisfies $\pi x \le \pi_0$ for some $i \in \{1, \ldots, m\}$. It follows that $\tilde{v}_i$ also satisfies $\pi x \le \pi_0$ because of (5). Therefore, $v_i$ satisfies $\pi x \le \pi_0$. Since $w_i \in P^{(1,1)}(m+1, \alpha, \epsilon)$, $(1 - \frac{\alpha}{m}, \ldots, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon)$ does not violate any 1-branch split cut that is generated from this 1-branch split disjunction because of (4).

Now we assume that $\tilde{e}_i$ satisfies $\pi x \ge \pi_0 + 1$ for every $1 \le i \le m$. Then $(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}) = \sum_{i=1}^m \frac{\tilde{e}_i}{m}$ satisfies $\pi x \ge \pi_0 + 1$. Since $(1 - \frac{\alpha}{m}, \ldots, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon) = \frac{\alpha}{m-1}(\frac{1}{m}, \ldots, \frac{1}{m}, \frac{(m-1)^2\epsilon}{m\alpha}) + (1 - \frac{\alpha}{m-1})(1, 1, \ldots, 1, 0)$ and $(\frac{1}{m}, \ldots, \frac{1}{m}, \frac{(m-1)^2\epsilon}{m\alpha}) \in P(m+1, \alpha, \epsilon)$ and $(1, 1, \ldots, 1, 0) \in P^{(1,1)}(m+1, \alpha, \epsilon)$ from Observation 1, the point $(1 - \frac{\alpha}{m}, \ldots, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon)$ does not violate any 1-branch split cut that is generated from this 1-branch split disjunction. This proves that $(1 - \frac{\alpha}{m}, \ldots, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon) \in P^{(1,1)}(m+1, \alpha, \epsilon)$.

$\subseteq$: We now show that $P^{(1,1)}(m+1, \alpha, \epsilon) \subseteq P(m+1, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon)$. To this end, we prove that the defining inequalities of $P(m+1, 1 - \frac{\alpha}{m}, \frac{m-1}{m}\epsilon)$, i.e.

$$\sum_{j=1}^m \frac{m-1}{m} \epsilon x_j + \alpha y \le (m-1)\epsilon, \tag{6}$$

$$(m-1)\epsilon x_j - (m-\alpha) y \ge 0, \quad \forall j = 1, \ldots, m, \tag{7}$$

are 1-branch split cuts.

**Claim 1.** *Inequality (6) is a 1-branch split cut generated by applying the 1-branch split disjunction* $(\sum_{j=1}^m x_j \le m-1) \vee (\sum_{j=1}^m x_j \ge m)$ *to* $P(m+1, \alpha, \epsilon)$.

**Proof of Claim 1.** We know that $P \equiv P(m+1, \alpha, \epsilon)$ is a $(m+1)$-dimensional simplex defined by $m+2$ constraints. For $i = 0, 1, \ldots, m+1$, we let $P_i$ be the polyhedral cone defined by the $m+1$ constraints of $P(m+1, \alpha, \epsilon)$ that are satisfied at equality at $\mathcal{V}_i$. We denote

$$C_P \equiv \text{Conv}\left(\left\{(x, y) \in P \,\middle|\, \sum_{j=1}^m x_j \le m-1\right\} \bigcup \left\{(x, y) \in P \,\middle|\, \sum_{j=1}^m x_j \ge m\right\}\right),$$

and

$$C_{P_i} \equiv \text{Conv}\left(\left\{(x, y) \in P_i \,\middle|\, \sum_{j=1}^m x_j \le m-1\right\} \bigcup \left\{(x, y) \in P_i \,\middle|\, \sum_{j=1}^m x_j \ge m\right\}\right),$$

for $i = 0, 1, \ldots, m+1$. It follows from the result in Section 1.2 of [1] that $C_P = \cap_{i=0}^{m+1} C_{P_i}$. By Lemma 1 of [1], the $m+1$ points at which the disjunctive plane $\sum_{j=1}^m x_j = m-1$ or $\sum_{j=1}^m x_j = m$ intersects with the $m+1$ half-lines starting at $\mathcal{V}_{m+1}$ in the directions of $\mathcal{V}_i - \mathcal{V}_{m+1}$, where $i = 0, 1, \ldots, m$, define a valid inequality for $C_{P_{m+1}}$. Because $C_P = \cap_{i=0}^{m+1} C_{P_i}$, this valid inequality is also valid for $C_P$. Therefore, it is a 1-branch split cut. In the following, we derive these $m+1$ intersection points and show that (6) is the resulting 1-branch split cut.

The vector $(\frac{m-1}{m}, \frac{m-1}{m}, \ldots, \frac{(m-1)\epsilon}{m\alpha})$ lies on the line segment between $\mathcal{V}_0$ and $\mathcal{V}_{m+1}$ and satisfies $\sum_{j=1}^m x_j = m-1$. The vectors $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ are the end points of the line segments between $\mathcal{V}_{m+1}$ and $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$, respectively. They satisfy $\sum_{j=1}^m x_j = m$. These $m+1$ vectors $(\frac{m-1}{m}, \frac{m-1}{m}, \ldots, \frac{(m-1)\epsilon}{m\alpha})$, $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_m$ all satisfy (6) at equality. Hence, Claim 1 is proved. □

**Claim 2.** *Inequality (7) is a 1-branch split cut generated by applying the 1-branch split disjunction* $(x_j \le 0) \vee (x_j \ge 1)$ *to* $P(m+1, \alpha, \epsilon)$*, where* $1 \le j \le m$.

**Proof of Claim 2.** Applying the idea used in the proof of Claim 1 again to the polyhedral cone $P_{m+1}$, we prove Claim 2 as follows.

Let $e_i$ be the unit vector in $\mathbb{R}^{m+1}$ whose $i$th component is 1 and whose other components are 0s. The vector $\sum_{i=1}^{j-1} \frac{(m-1)\alpha}{m-\alpha} e_i + e_j + \sum_{i=j+1}^{m} \frac{(m-1)\alpha}{m-\alpha} e_i + \frac{(m-1)\epsilon}{m-\alpha} e_{m+1}$ lies on the line segment between $\mathcal{V}_{m+1}$ and $\mathcal{V}_j$ and satisfies $x_j = 1$. The vectors $\mathcal{V}_0$ and $\mathcal{V}_i$, where $1 \le i \le j-1$ and $j+1 \le i \le m$, are the end points of the line segments between these points and $\mathcal{V}_{m+1}$, respectively. They satisfy $x_j = 0$. Further, these $m+1$ vectors, $\sum_{i=1}^{j-1} \frac{(m-1)\alpha}{m-\alpha} e_i + e_j + \sum_{i=j+1}^{m} \frac{(m-1)\alpha}{m-\alpha} e_i + \frac{(m-1)\epsilon}{m-\alpha} e_{m+1}$, $\mathcal{V}_0$ and $\mathcal{V}_i$, where $1 \le i \le j-1$ and $j+1 \le i \le m$, all satisfy (7) at equality. Hence, Claim 2 is proved. □

We now use the result of Proposition 1 to inductively obtain the $k$th 1-branch split closure of $P(m+1, \frac{m-1}{m}, \epsilon)$.

**Corollary 2.** *Assume $m \ge 2$ and $0 < \epsilon < 1$. Then $P^{(k,1)}(m+1, \frac{m-1}{m}, \epsilon) = P(m+1, a_k, b_k)$ where*

$$a_k = \begin{cases} \dfrac{m}{m+1} + \dfrac{1}{(m+1)m^{k+1}} & \text{if } k \text{ is odd,} \\ \dfrac{m}{m+1} - \dfrac{1}{(m+1)m^{k+1}} & \text{if } k \text{ is even,} \end{cases}$$

*and*

$$b_k = \left(\frac{m-1}{m}\right)^k \epsilon.$$

**Proof.** The proof is by induction. When $k = 0$, the result follows from the definition of $P(m+1, \frac{m-1}{m}, \epsilon)$. Assume that the result holds for $k = 2i$, we will prove it for $k = 2i+1$ and $2i+2$.

Since $P^{(2i,1)}(m+1, \frac{m-1}{m}, \epsilon) = P(m+1, a_{2i}, b_{2i})$, by Proposition 1 we have $P^{(2i+1,1)}(m+1, \frac{m-1}{m}, \epsilon) = P^{(1,1)}(m+1, a_{2i}, b_{2i}) = P(m+1, 1-\frac{a_{2i}}{m}, \frac{m-1}{m} b_{2i}) = P(m+1, 1-\frac{1}{m}(\frac{m}{m+1} - \frac{1}{(m+1)m^{2i+1}}), \frac{m-1}{m}(\frac{m-1}{m})^{2i}\epsilon) = P(m+1, \frac{m}{m+1} + \frac{1}{(m+1)m^{2i+2}}, (\frac{m-1}{m})^{2i+1}\epsilon) = P(m+1, a_{2i+1}, b_{2i+1})$.

Since $P^{(2i+1,1)}(m+1, \frac{m-1}{m}, \epsilon) = P(m+1, a_{2i+1}, b_{2i+1})$, by Proposition 1 we have $P^{(2i+2,1)}(m+1, \frac{m-1}{m}, \epsilon) = P^{(1,1)}(m+1, a_{2i+1}, b_{2i+1}) = P(m+1, 1-\frac{a_{2i+1}}{m}, \frac{m-1}{m} b_{2i+1}) = P(m+1, 1-\frac{1}{m}(\frac{m}{m+1} + \frac{1}{(m+1)m^{2i+2}}), \frac{m-1}{m}(\frac{m-1}{m})^{2i+1}\epsilon) = P(m+1, \frac{m}{m+1} - \frac{1}{(m+1)m^{2i+3}}, (\frac{m-1}{m})^{2i+2}\epsilon) = P(m+1, a_{2i+2}, b_{2i+2})$. □

## 3. Multi-branch split ranks

In this section, we first show that the $m$-branch split rank of $P(m+1, \frac{m-1}{m}, \epsilon)$ is 1 for $m \ge 2$. Then, we prove that the 2-branch split rank of $P(m+1, \frac{m-1}{m}, \epsilon)$ is infinite for $m \ge 3$. This result generalizes, to the 2-branch split rank, the fact first observed by Cook, Kannan and Schrijver that the 1-branch split rank of valid inequalities of mixed integer programs can be infinite. We then pose two conjectures: the first is on the $t$-branch split rank of $P(m+1, \frac{m-1}{m}, \epsilon)$ for $m \ge 2$ and $1 \le t \le m-1$, and the second is on the polyhedrality of the 1st $t$-branch split closure ($t \ge 2$) of the general polyhedron $P$ defined in Section 1.

**Proposition 3.** *The $m$-branch split rank of $P(m+1, \frac{m-1}{m}, \epsilon)$ is 1 for $m \ge 2$.*

**Proof.** Given an integer $m \ge 2$, we prove that the $m$-branch split rank of $P(m+1, \frac{m-1}{m}, \epsilon)$ is 1 by showing that $y \le 0$ is a valid inequality for all $\{(x, y) \in P(m+1, \frac{m-1}{m}, \epsilon) \mid x_j \le 0 \text{ for } j \in S; x_j \ge 1 \text{ for } j \notin S\}$, where $S$ is an arbitrary subset of $\{1, \dots, m\}$.

Consider an arbitrary subset $S \subseteq \{1, \dots, m\}$. Assuming $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{y}) \in \{(x, y) \in P(m+1, \frac{m-1}{m}, \epsilon) \mid x_j \le 0 \text{ for } j \in S; x_j \ge 1 \text{ for } j \notin S\}$, we will prove $\tilde{y} = 0$ by contradiction.

Assume that $\tilde{y} > 0$. Then $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{y}) = \beta_0(0, 0, \dots, 0) + \beta_1(m, 0, \dots, 0) + \beta_2(0, m, 0, \dots, 0) + \dots + \beta_m(0, 0, \dots, m, 0) + \beta_{m+1}(\frac{m-1}{m}, \frac{m-1}{m}, \dots, \frac{m-1}{m}, \epsilon)$, where $\sum_{i=0}^{m+1} \beta_i = 1$, $\beta_i \ge 0$ for $0 \le i \le m$, and $\beta_{m+1} > 0$. Hence, $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{y}) = (\beta_1 m + \beta_{m+1}\frac{m-1}{m}, \beta_2 m + \beta_{m+1}\frac{m-1}{m}, \dots, \beta_m m + \beta_{m+1}\frac{m-1}{m}, \beta_{m+1}\epsilon)$. Since $(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{y}) \in \{(x, y) \in P(m+1, \frac{m-1}{m}, \epsilon) \mid x_j \le 0 \text{ for } j \in S; x_j \ge 1 \text{ for } j \notin S\}$ and $\beta_{m+1} > 0$, we must have $S = \emptyset$. Therefore, $\beta_i m + \beta_{m+1}\frac{m-1}{m} \ge 1$ for $1 \le i \le m$. Adding them up, we obtain $m \sum_{i=1}^{m} \beta_i + \beta_{m+1}(m-1) \ge m$, *i.e.*, $(1-\beta_0)m - \beta_{m+1} \ge m$, which is a contradiction to the fact that $\beta_0 \ge 0$ and $\beta_{m+1} > 0$. □

Proposition 3 establishes that multi-branch split disjunctions can generate cutting planes that are significantly stronger than those generated using single-branch split disjunctions. However, we will show in the following propositions that multi-branch split cuts may not always be sufficient to obtain the integral hull of a polyhedron, even if a finite number of rounds is applied.

**Proposition 4.** *The* 2*-branch split rank of* $P(m+1, \frac{m-1}{m}, \epsilon)$ *is infinite for* $m \geq 3$. □

We provide a proof of Proposition 4 in two steps. First, in Lemma 5, we show that the result holds when $m = 3$. Then we prove in Lemma 6 that the result also holds when $m \geq 4$.

**Lemma 5.** *The* 2*-branch split rank of* $P(4, \frac{2}{3}, \epsilon)$ *is infinite.*

**Proof.** To prove the result, we show that the vector $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2\epsilon}{75})$ is in the 1st 2-branch split closure of $P(4, \frac{2}{3}, \epsilon)$. More specifically, given any 2-branch split disjunction:

$$\begin{aligned}&(\pi_1^1 x_1 + \pi_2^1 x_2 + \pi_3^1 x_3 \leq \pi_0^1, \pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 \leq \pi_0^2)\\ &\vee (\pi_1^1 x_1 + \pi_2^1 x_2 + \pi_3^1 x_3 \leq \pi_0^1, \pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 \geq \pi_0^2 + 1)\\ &\vee (\pi_1^1 x_1 + \pi_2^1 x_2 + \pi_3^1 x_3 \geq \pi_0^1 + 1, \pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 \leq \pi_0^2)\\ &\vee (\pi_1^1 x_1 + \pi_2^1 x_2 + \pi_3^1 x_3 \geq \pi_0^1 + 1, \pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 \geq \pi_0^2 + 1),\end{aligned}$$

we prove that the vector $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2\epsilon}{75})$ satisfies all 2-branch split cuts generated from this disjunction.

If a 2-branch split cut that is generated from the above disjunction cuts off some point from $P(4, \frac{2}{3}, \epsilon)$, it must cut off the point $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \epsilon)$. So, without loss of generality, we may assume that $\pi_1^1, \pi_2^1, \pi_3^1$ and $\pi_0^1$ satisfy:

$$\pi_0^1 < \frac{2}{3}(\pi_1^1 + \pi_2^1 + \pi_3^1) < \pi_0^1 + 1. \tag{8}$$

Otherwise, $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \epsilon)$ satisfies the above 2-branch split disjunction, say $\frac{2}{3}(\pi_1^1+\pi_2^1+\pi_3^1) \leq \pi_0^1$ and $\frac{2}{3}(\pi_1^2+\pi_2^2+\pi_3^2) \leq \pi_0^2$, then the point $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \epsilon)$ will not be cut off by any 2-branch split cut that is generated from the 2-branch split disjunction.

Let $v = (1, 1, 1, 0)$, $v_1 = (0, 1, 1, 0)$, $v_2 = (1, 0, 1, 0)$ and $v_3 = (1, 1, 0, 0)$. We know that $v$, $v_1$, $v_2$ and $v_3$ are in $P(4, \frac{2}{3}, \epsilon)$. Without loss of generality, we assume that $v$ satisfies

$$\pi_1^1 x_1 + \pi_2^1 x_2 + \pi_3^1 x_3 \geq \pi_0^1 + 1. \tag{9}$$

Now, we claim that at least one of $v_1$, $v_2$ and $v_3$ also satisfies (9). Otherwise, all of $v_1, v_2, v_3$ would satisfy the constraint

$$\pi_1^1 x_1 + \pi_2^1 x_2 + \pi_3^1 x_3 \leq \pi_0^1 \tag{10}$$

and so, their convex combination under equal weights of $\frac{1}{3}$ would satisfy (10), which is a contradiction to (8).

We therefore assume that $v_1$ satisfies (9). This assumption is without loss of generality since integer variables can be reordered. We now consider two cases:

*Case* 1. At least one of the vectors $v_2$ and $v_3$ satisfies (9). Without loss of generality, we assume that $v_2$ satisfies (9). Then, by the pigeonhole principle, at least two of the vectors $v$, $v_1$ and $v_2$ must satisfy either

$$\pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 \geq \pi_0^2 + 1 \tag{11}$$

or

$$\pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 \leq \pi_0^2. \tag{12}$$

Without loss of generality, we assume that at least two of $v$, $v_1$ and $v_2$ satisfy (12). It is sufficient to consider the following two situations:

*Case* 1.1. $v$ and $v_1$ satisfy (12). Then the convex combination of $v$ and $v_1$ under equal weights of $\frac{1}{2}$ would satisfy (12), *i.e.*, $(\frac{1}{2}, 1, 1, 0)$ satisfies (12). Therefore, $(\frac{1}{2}, 1, 1, \frac{\epsilon}{2})$ satisfies (12). Also we know that $(\frac{1}{2}, 1, 1, \frac{\epsilon}{2})$ satisfies (9) because $v$ and $v_1$ satisfy (9). Further, it can be verified that this point belongs to $P(4, \frac{2}{3}, \epsilon)$. Since $(1, 0, 0, 0)$ must be in the 2-branch split closure (because of Observation 1), the convex combination of $(\frac{1}{2}, 1, 1, \frac{\epsilon}{2})$ and $(1, 0, 0, 0)$ under weights $\frac{2}{3}$ and $\frac{1}{3}$ is not cut off by any of the 2-branch split cuts generated by the given 2-branch split disjunction, i.e., $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{\epsilon}{3})$ satisfies all the 2-branch split cuts generated by the given 2-branch split disjunction. Using Observation 2, it follows that $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2\epsilon}{75})$ satisfies all the 2-branch split cuts generated by the given 2-branch split disjunction.

*Case* 1.2. $v_1$ and $v_2$ satisfy (12). Then the convex combination of $v_1$ and $v_2$ under equal weights $\frac{1}{2}$ would satisfy (12), *i.e.*, $(\frac{1}{2}, \frac{1}{2}, 1, 0)$ satisfies (12). Hence $(\frac{1}{2}, \frac{1}{2}, 1, \frac{3\epsilon}{4})$ satisfies (12). Also $(\frac{1}{2}, \frac{1}{2}, 1, \frac{3\epsilon}{4})$ satisfies (9), because $v_1$ and $v_2$ satisfy (9). Further, it can be verified that this point belongs to $P(4, \frac{2}{3}, \epsilon)$. Since $(1, 1, 0, 0)$ is in the 2-branch split closure (because of Observation 1), the convex combination of $(\frac{1}{2}, \frac{1}{2}, 1, \frac{3\epsilon}{4})$ and $(1, 1, 0, 0)$ under weights $\frac{2}{3}$ and $\frac{1}{3}$ is not cut off by any of the 2-branch split cuts generated by the given 2-branch split disjunction, i.e., $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{\epsilon}{2})$ satisfies all the 2-branch split cuts generated by the given 2-branch split disjunction. Using Observation 2, it follows that $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2\epsilon}{75})$ satisfies all the 2-branch split cuts generated by the given 2-branch split disjunction.

*Case* 2. Both $v_2$ and $v_3$ satisfy (10). Since $v$ and $v_1$ satisfy (9), $a \equiv (\frac{1}{2}, 1, 1, 0) = \frac{1}{2}v + \frac{1}{2}v_1$ satisfies (9). Similarly, since $v_2$ and $v_3$ satisfy (10), $b \equiv (1, \frac{1}{2}, \frac{1}{2}, 0)$ satisfies (10). It can be easily verified that $a$ and $b$ belong to $P(4, \frac{2}{3}, \epsilon)$. Now we define the following vectors:

$$a_1 \equiv \frac{4}{5}(3,0,0,0) + \frac{1}{5}a = \left(\frac{5}{2}, \frac{1}{5}, \frac{1}{5}, 0\right),$$
$$b_1 \equiv \frac{3}{4}(3,0,0,0) + \frac{1}{4}b = \left(\frac{5}{2}, \frac{1}{8}, \frac{1}{8}, 0\right),$$
$$a_2 \equiv \frac{3}{4}(0,3,0,0) + \frac{1}{4}a = \left(\frac{1}{8}, \frac{5}{2}, \frac{1}{4}, 0\right),$$
$$b_2 \equiv \frac{4}{5}(0,3,0,0) + \frac{1}{5}b = \left(\frac{1}{5}, \frac{5}{2}, \frac{1}{10}, 0\right),$$
$$a_3 \equiv \frac{3}{4}(0,0,3,0) + \frac{1}{4}a = \left(\frac{1}{8}, \frac{1}{4}, \frac{5}{2}, 0\right),$$
$$b_3 \equiv \frac{4}{5}(0,0,3,0) + \frac{1}{5}b = \left(\frac{1}{5}, \frac{1}{10}, \frac{5}{2}, 0\right),$$
$$a_4 \equiv \frac{3}{5}(0,0,0,0) + \frac{2}{5}a = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, 0\right),$$
$$b_4 \equiv \frac{1}{2}(0,0,0,0) + \frac{1}{2}b = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, 0\right).$$

Since each of points $(3, 0, 0, 0)$, $(0, 3, 0, 0)$, $(0, 0, 3, 0)$ and $(0, 0, 0, 0)$ satisfies (9) or (10), since $a$ satisfies (9) and since $b$ satisfies (10), then at least one of $a_i$ and $b_i$ satisfies (9) or (10) for $i = 1, 2, 3, 4$. Define now the following vectors:

$$\bar{a}_1 \equiv \left(\frac{5}{2}, \frac{1}{5}, \frac{1}{5}, \frac{\epsilon}{10}\right), \qquad \bar{b}_1 \equiv \left(\frac{5}{2}, \frac{1}{8}, \frac{1}{8}, \frac{3\epsilon}{16}\right),$$
$$\bar{a}_2 \equiv \left(\frac{1}{8}, \frac{5}{2}, \frac{1}{4}, \frac{\epsilon}{8}\right), \qquad \bar{b}_2 \equiv \left(\frac{1}{5}, \frac{5}{2}, \frac{1}{10}, \frac{3\epsilon}{20}\right),$$
$$\bar{a}_3 \equiv \left(\frac{1}{8}, \frac{1}{4}, \frac{5}{2}, \frac{\epsilon}{8}\right), \qquad \bar{b}_3 \equiv \left(\frac{1}{5}, \frac{1}{10}, \frac{5}{2}, \frac{3\epsilon}{20}\right),$$
$$\bar{a}_4 \equiv \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3\epsilon}{10}\right), \qquad \bar{b}_4 \equiv \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{3\epsilon}{8}\right).$$

A direct verification shows that all of the $\bar{a}_i s$ and $\bar{b}_i s$ belong to $P(4, \frac{2}{3}, \epsilon)$ for $i = 1, 2, 3, 4$. Now we see that at least one of $\bar{a}_i$ and $\bar{b}_i$ satisfies (9) or (10) for $i = 1, 2, 3, 4$. For $i = 1, 2, 3, 4$, we denote by $\bar{c}_i$ the vector among $\bar{a}_i$ and $\bar{b}_i$ that satisfies (9) or (10).

**Claim.** *At least one of $\bar{c}_1$, $\bar{c}_2$, $\bar{c}_3$ and $\bar{c}_4$ satisfies (11) or (12).*

**Proof of Claim.** Without loss of generality, we assume that $\pi_0^2 \geq 0$. Assuming that $\bar{c}_1$, $\bar{c}_2$ and $\bar{c}_3$ satisfy $\pi_0^2 < \pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 < \pi_0^2 + 1$, we will prove that $\bar{c}_4$ must satisfy (12).

First, we show that $\pi_0^2 \geq 1$. If $\pi_0^2 = 0$, then $\bar{c}_1$, $\bar{c}_2$, and $\bar{c}_3$ all satisfy $0 < \pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 < 1$. Without loss of generality, we assume that $\pi_1^2$ is the one among $\pi_1^2$, $\pi_2^2$ and $\pi_3^2$ that has the largest absolute value. Since $\bar{c}_1$ satisfies $0 < \pi_1^2 x_1 + \pi_2^2 x_2 + \pi_3^2 x_3 < 1$ and $\bar{c}_1$ is one of $\bar{a}_1$ and $\bar{b}_1$, we obtain a contradiction. Hence, $\pi_0^2 \geq 1$.

Second, from $\pi_0^2 < \pi^2 \bar{c}_i < \pi_0^2 + 1$ for $i = 1, 2, 3$, we show that there exists $\alpha \in (0, \frac{1}{2})$ such that $\alpha \pi_0^2 < \pi^2 \bar{c}_4 < \alpha(\pi_0^2 + 1)$. To this end, we consider each of the 16 cases that are obtained by assigning $\bar{c}_i$ to one of the two values $\bar{a}_i$ or $\bar{b}_i$ for $i = 1, 2, 3, 4$. Consider first the case where $\bar{c}_1 = \bar{b}_1$, $\bar{c}_2 = \bar{b}_2$, $\bar{c}_3 = \bar{b}_3$ and $\bar{c}_4 = \bar{a}_4$. Since $\pi_0^2 < \pi^2 \bar{b}_1 < \pi_0^2 + 1$ and $\pi_0^2 < \pi^2 \bar{b}_2 < \pi_0^2 + 1$ and $\pi_0^2 < \pi^2 \bar{b}_3 < \pi_0^2 + 1$, one can easily verify that the linear combination of these inequalities under the positive multipliers $\psi_1 = 24/430$, $\psi_2 = 65/430$ and $\psi_3 = 65/430$ gives $(\psi_1 + \psi_2 + \psi_3)\pi_0^2 < \pi^2 \bar{a}_4 = \frac{1}{5}\pi_1^2 + \frac{2}{5}\pi_2^2 + \frac{2}{5}\pi_3^2 < (\psi_1 + \psi_2 + \psi_3)(\pi_0^2 + 1)$. Now defining $\alpha \equiv \psi_1 + \psi_2 + \psi_3$ we obtain that $\alpha = \frac{154}{430} < \frac{1}{2}$ as desired. Proofs for the remaining cases are obtained analogously by selecting the positive multipliers $\psi_1$, $\psi_2$ and $\psi_3$ as given in Table 1 of the Appendix.

Now, because $\alpha < \frac{1}{2}$ and $\pi_0^2 \geq 1$, we obtain that $\alpha(\pi_0^2 + 1) < \pi_0^2$. Hence, $\pi^2 \bar{c}_4 < \pi_0^2$, *i.e.*, $\bar{c}_4$ satisfies (12). This concludes the proof of the claim. □

By the claim, we know that at least one of $\bar{a}_1, \bar{b}_1, \bar{a}_2, \bar{b}_2, \bar{a}_3, \bar{b}_3, \bar{a}_4, \bar{b}_4$ satisfies the 2-branch split disjunction. Consider now the following identities:

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2\epsilon}{75}\right) = \frac{4}{15}\bar{a}_1 + \frac{11}{15}\left(0, \frac{46}{55}, \frac{46}{55}, 0\right), \tag{13}$$

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{\epsilon}{20}\right) = \frac{4}{15}\bar{b}_1 + \frac{11}{15}\left(0, \frac{19}{22}, \frac{19}{22}, 0\right), \tag{14}$$

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{\epsilon}{30}\right) = \frac{4}{15}\bar{a}_2 + \frac{11}{15}\left(\frac{19}{22}, 0, \frac{9}{11}, 0\right), \tag{15}$$

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{\epsilon}{25}\right) = \frac{4}{15}\bar{b}_2 + \frac{11}{15}\left(\frac{46}{55}, 0, \frac{48}{55}, 0\right), \tag{16}$$

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{\epsilon}{30}\right) = \frac{4}{15}\bar{a}_3 + \frac{11}{15}\left(\frac{19}{22}, \frac{9}{11}, 0, 0\right), \tag{17}$$

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{\epsilon}{25}\right) = \frac{4}{15}\bar{b}_3 + \frac{11}{15}\left(\frac{46}{55}, \frac{48}{55}, 0, 0\right), \tag{18}$$

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3\epsilon}{20}\right) = \frac{1}{2}\bar{a}_4 + \frac{1}{2}\left(\frac{17}{15}, \frac{14}{15}, \frac{14}{15}, 0\right), \tag{19}$$

$$\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{3\epsilon}{16}\right) = \frac{1}{2}\bar{b}_4 + \frac{1}{2}\left(\frac{5}{6}, \frac{13}{12}, \frac{13}{12}, 0\right). \tag{20}$$

In each of the above identities, the vector on the left is a convex combination of a vector $\bar{a}_i$ or $\bar{b}_i$ and another vector that belongs to $P_I(4, \frac{2}{3}, \epsilon)$, where $P_I(4, \frac{2}{3}, \epsilon)$ is the integral hull of $P(4, \frac{2}{3}, \epsilon)$. One of these eight vectors must belong to the 1st 2-branch split closure. We therefore conclude that the vector $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{2\epsilon}{75})$ is in the 1st 2-branch split closure because of Observation 2. $\square$

**Lemma 6.** *The* 2*-branch split rank of* $P(m+1, \frac{m-1}{m}, \epsilon)$ *is infinite for* $m \geq 4$.

**Proof.** To prove this result, we show that the vector $(\frac{m-1}{m}, \frac{m-1}{m}, \ldots, \frac{m-1}{m}, \frac{\epsilon}{m})$ is in the 1st 2-branch split closure of $P(m+1, \frac{m-1}{m}, \epsilon)$. Consider any 2-branch split disjunction:

$$\left(\sum_{j=1}^{m}\pi_j^1 x_j \leq \pi_0^1, \sum_{j=1}^{m}\pi_j^2 x_j \leq \pi_0^2\right) \vee \left(\sum_{j=1}^{m}\pi_j^1 x_j \leq \pi_0^1, \sum_{j=1}^{m}\pi_j^2 x_j \geq \pi_0^2 + 1\right)$$
$$\vee \left(\sum_{j=1}^{m}\pi_j^1 x_j \geq \pi_0^1 + 1, \sum_{j=1}^{m}\pi_j^2 x_j \leq \pi_0^2\right) \vee \left(\sum_{j=1}^{m}\pi_j^1 x_j \geq \pi_0^1 + 1, \sum_{j=1}^{m}\pi_j^2 x_j \geq \pi_0^2 + 1\right).$$

We next prove that the vector $(\frac{m-1}{m}, \frac{m-1}{m}, \ldots, \frac{m-1}{m}, \frac{\epsilon}{m})$ satisfies all 2-branch split cuts generated from this 2-branch split disjunction.

Analogous to the proof of Lemma 5, we define $v = (1, 1, 1, \ldots, 1, 0)$, $v_1 = (0, 1, 1, \ldots, 1, 0)$, $v_2 = (1, 0, 1, \ldots, 1, 0)$, $v_3 = (1, 1, 0, 1, \ldots, 1, 0), \ldots, v_m = (1, 1, \ldots, 1, 0, 0)$. It can easily be verified that $v, v_1, v_2, \ldots, v_m$ are in $P(m+1, \frac{m-1}{m}, \epsilon)$.

Because $m \geq 4$, it follows from the pigeonhole principle that there are at least two vectors of $v, v_1, v_2, \ldots, v_m$ that satisfy:

$$\left(\sum_{j=1}^{m}\pi_j^1 x_j \leq \pi_0^1, \sum_{j=1}^{m}\pi_j^2 x_j \leq \pi_0^2\right) \quad \text{or} \quad \left(\sum_{j=1}^{m}\pi_j^1 x_j \leq \pi_0^1, \sum_{j=1}^{m}\pi_j^2 x_j \geq \pi_0^2 + 1\right) \text{ or}$$
$$\left(\sum_{j=1}^{m}\pi_j^1 x_j \geq \pi_0^1 + 1, \sum_{j=1}^{m}\pi_j^2 x_j \leq \pi_0^2\right) \quad \text{or} \quad \left(\sum_{j=1}^{m}\pi_j^1 x_j \geq \pi_0^1 + 1, \sum_{j=1}^{m}\pi_j^2 x_j \geq \pi_0^2 + 1\right).$$

Without loss of generality, we assume that two vectors of $v, v_1, v_2, \ldots, v_m$ satisfy

$$\sum_{j=1}^{m}\pi_j^1 x_j \leq \pi_0^1 \quad \text{and} \quad \sum_{j=1}^{m}\pi_j^2 x_j \leq \pi_0^2. \tag{21}$$

Since integer variables can be reordered, it is sufficient to consider the following two situations:

*Case* 1. $v$ and $v_1$ satisfy (21). Then their convex combination under equal weight of $\frac{1}{2}$ satisfies (21), *i.e.*, $(\frac{1}{2}, 1, 1, \ldots, 1, 0)$ satisfies (21). Therefore, the vector $(\frac{1}{2}, 1, 1, \ldots, 1, \frac{\epsilon}{2})$ in $P(m+1, \frac{m-1}{m}, \epsilon)$ also satisfies (21). Since $(1, 0, 0, \ldots, 0)$ must be in the 2-branch split closure (because of Observation 1), the convex combination of $(\frac{1}{2}, 1, 1, \ldots, 1, \frac{\epsilon}{2})$ and $(1, 0, 0, \ldots, 0)$ under weights $\frac{2}{3}$ and $\frac{1}{3}$ must satisfy all the 2-branch split cuts generated by the 2-branch split disjunction, namely, $(\frac{2}{3}, \frac{2}{3}, \ldots, \frac{2}{3}, \frac{\epsilon}{3})$ satisfies all the 2-branch split cuts generated by the 2-branch split disjunction. Since $(1, 1, \ldots, 1, 0)$ is in the 2-branch split closure, the vector $(\frac{m-1}{m}, \frac{m-1}{m}, \ldots, \frac{m-1}{m}, \frac{\epsilon}{m})$ as a convex combination of $(\frac{2}{3}, \frac{2}{3}, \ldots, \frac{2}{3}, \frac{\epsilon}{3})$ and $(1, 1, \ldots, 1, 0)$ under weights $\frac{3}{m}$ and $\frac{m-3}{m}$ must satisfy all the 2-branch split cuts generated by the 2-branch split disjunction.

*Case* 2. $v_1$ and $v_2$ satisfy (21). Then their convex combination under equal weights $\frac{1}{2}$ satisfies (21), *i.e.*, $(\frac{1}{2}, \frac{1}{2}, 1, \ldots, 1, 0)$ satisfies (21). Therefore, the vector $(\frac{1}{2}, \frac{1}{2}, 1, \ldots, 1, \frac{m\epsilon}{2(m-1)})$ in $P(m+1, \frac{m-1}{m}, \epsilon)$ also satisfies (21). Since $(1, 1, 0, \ldots, 0)$ is in the 2-branch split closure (because of Observation 1), the convex combination of $(\frac{1}{2}, \frac{1}{2}, 1, \ldots, \frac{m\epsilon}{2(m-1)})$ and $(1, 1, 0, \ldots, 0)$ under weights $\frac{2}{3}$ and $\frac{1}{3}$ satisfies all the 2-branch split cuts generated by the 2-branch split disjunction, namely, $(\frac{2}{3}, \frac{2}{3}, \ldots, \frac{2}{3}, \frac{m\epsilon}{3(m-1)})$ satisfies all the 2-branch split cuts generated by the 2-branch split disjunction. Since $(1, 1, \ldots, 1, 0)$ is in the 2-branch split closure, the vector $(\frac{m-1}{m}, \frac{m-1}{m}, \ldots, \frac{m-1}{m}, \frac{\epsilon}{m-1})$ as a convex combination of $(\frac{2}{3}, \frac{2}{3}, \ldots, \frac{2}{3}, \frac{m\epsilon}{3(m-1)})$ and $(1, 1, \ldots, 1, 0)$ under weights $\frac{3}{m}$ and $\frac{m-3}{m}$ must also satisfy all the 2-branch split cuts generated by the 2-branch split disjunction. Hence, $(\frac{m-1}{m}, \frac{m-1}{m}, \ldots, \frac{m-1}{m}, \frac{\epsilon}{m})$ satisfies all the 2-branch split cuts generated by the 2-branch split disjunction. □

In light of the result of Proposition 4, we are proposing the following conjecture about the $t$-branch split rank of $P(m+1, \frac{m-1}{m}, \epsilon)$ for general $t$:

**Conjecture 7.** *The $t$-branch split rank of $P(m+1, \frac{m-1}{m}, \epsilon)$ is infinite for $m \geq t+1$, where $t$ is a positive integer.*

Although we do not know of a proof for this conjecture, a weaker result can easily be derived from Lemma 6. In fact, in the proof of Lemma 6, we use the assumption $m \geq 4$ to invoke the pigeonhole principle and argue that one of the four disjunctive terms contains at least two of the points $v, v_1, \ldots, v_m$. It is easy to see, in the case of a $t$-branch disjunction, the proof of Lemma 6 remains valid if one of the $2^t$ disjunctive terms contains at least two of the points $v, v_1, \ldots, v_m$. Hence, we have the following result.

**Proposition 8.** *The $t$-branch split rank of $P(m+1, \frac{m-1}{m}, \epsilon)$ is infinite for $m \geq 2^t$, where $t$ is a positive integer.* □

Proposition 8 establishes that, as the dimension of the set considered becomes large, even split disjunctions with a large number of branches might not be sufficient to derive certain valid inequalities in a finite number of rounds.

Cook, Kannan and Schrijver [5] proved that the 1st 1-branch split closure of a polyhedron $P$ is again a polyhedron. They also showed that the 1st 1-branch split closure of $P$ can be constructed by a finite number of 1-branch split cuts that are generated using only 1-branch split disjunctions with bounded coefficients. If only $t$-branch split disjunctions with bounded coefficients are needed to construct the $t$-branch split closure of $P$, then the following conjecture would be easily proven.

**Conjecture 9.** *The* 1st *$t$-branch split closure ($t \geq 2$) of a polyhedron $P$, which is the continuous relaxation of a mixed integer set, is a polyhedron.* □

Conjectures 7 and 9 are related. In particular, if one could prove that only bounded $t$-branch split disjunctions are needed to construct the 1st $t$-branch split closure of a polyhedron $P$, then showing that $y \leq 0$ is not valid for

$$\operatorname{conv}\left(\bigcup_{S\subseteq\{1,\ldots,t\}}\left\{(x,y)\in P\left(m+1,\frac{m-1}{m},\epsilon\right)\middle|\ \pi^i x\leq\pi_0^i \text{ for } i\in S;\ \pi^i x\geq\pi_0^i+1 \text{ for } i\notin S\right\}\right),$$

for any given integral $\{\pi^1, \pi^2, \ldots, \pi^t\}$ and $\{\pi_0^1, \pi_0^2, \ldots, \pi_0^t\}$ would give a direct proof of Conjecture 7.

We conclude this section by pointing out that the polytope $P(m+1, \frac{m-1}{m}, \epsilon)$ can be extended to an unbounded polyhedron if the nonnegativity constraint on $y$ is removed from the inequality formulation of $P(m+1, \frac{m-1}{m}, \epsilon)$. As a result, all the integer variables are now unbounded. Since $\epsilon$ is a rational number less than one, it is not hard to see that $y \leq 0$ is the only additional inequality needed to describe the integral hull of the unbounded polyhedron. For this family of unbounded polyhedra, it is easily verified that the results derived earlier still apply. Further, the unboundedness of integer variables makes it impossible to use binary expansions to reformulate the unbounded polyhedron with a finite number of binary variables. Therefore, the finite rank of the lift-and-project cuts on mixed 0-1 polyhedra cannot be directly applied.

## 4. Some related results

In this section, we describe different ways of obtaining $y \le 0$, the inequality of $P_I(m+1, \frac{m-1}{m}, \epsilon)$ that seems difficult to obtain from split disjunctions. First, we show in Proposition 10 that it is possible to obtain the integer hull $P_I(m+1, \frac{m-1}{m}, \epsilon)$ using a "nonlinear" 1-branch split disjunction. We then mention different research directions that have been investigated to tackle the same problem. In particular, we discuss the work of Owen and Mehrotra [8] and Andersen, Louveaux, Weismantel and Wolsey [2].

It follows from our results in previous sections that the inequality $y < 0$ cannot be obtained easily if the number of branches in the split disjunction is small in comparison to the dimension of the problem. It also follows from Proposition 3 that $y \le 0$ can be easily generated if a $m$-branch split disjunction is used on the polytope $P(m+1, \frac{m-1}{m}, \epsilon)$. We show next that, provided that "nonlinear" disjunctions are used, a single disjunction is sufficient to obtain $y \le 0$.

Given $\pi_0 \in \mathbb{Z}$ and a function $\pi : \mathbb{R}^n \to \mathbb{R}$ that satisfies $\pi(x) \in \mathbb{Z}, \forall x \in \mathbb{Z}^n$, we consider

$$(\pi(x) \le \pi_0) \vee (\pi(x) \ge \pi_0 + 1), \tag{22}$$

which we refer to as a *nonlinear split disjunction*. Clearly nonlinear split disjunctions generalize traditional split disjunctions as $\pi(.)$ can be chosen to be a linear function with integer coefficients. Further, the set of nonlinear split disjunctions can be large, depending on what nonlinear functions are allowed when defining the disjunction. The next proposition shows that nonlinear split disjunctions can be significantly stronger than linear ones.

**Proposition 10.** *The inequality $y \le 0$ is valid for $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \left(\{\prod_{j=1}^m x_j \le 0\} \cup \{\prod_{j=1}^m x_j \ge 1\}\right)$, where $m \ge 2$.*

**Proof.** We will prove that $y \le 0$ is valid for $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \le 0\}$ and $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \ge 1\}$. We refer to the inequality formulation of $P(m+1, \frac{m-1}{m}, \epsilon)$ given in Section 2.

First, we show that $y \le 0$ is valid for $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \le 0\}$. Since $x \in \mathbb{R}^m_+$, every vector in $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \le 0\}$ has some $x_j = 0$. Then the constraints $\epsilon x_j - \frac{m-1}{m} y \ge 0, \forall j = 1, \dots, m$, imply that $y = 0$ for every vector in $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \le 0\}$.

Second, we show that $y \le 0$ is valid for $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \ge 1\}$. Because of the constraint $\sum_{j=1}^m \epsilon x_j + y \le m\epsilon$, every vector in $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \ge 1\}$ satisfies $\sum_{j=1}^m x_j \le m$, *i.e.*, $\sum_{j=1}^m \frac{x_j}{m} \le 1$. By the well-known Arithmetic Mean Geometric Mean (AM-GM) inequality, every vector in $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \ge 1\}$ satisfies:

$$\prod_{j=1}^m \frac{x_j}{m} \le \left( \frac{\sum_{j=1}^m \frac{x_j}{m}}{m} \right)^m .$$

Therefore, $1 \le \prod_{j=1}^m x_j \le (\sum_{j=1}^m \frac{x_j}{m})^m \le 1$. It follows from the AM-GM result that equality holds if and only if $x_1 = x_2 = \dots = x_m = 1$. Now we know that the only vector in $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \ge 1\}$ is $(1, 1, \dots, 1, 0)$. Thus, $y \le 0$ is a valid inequality for $P(m+1, \frac{m-1}{m}, \epsilon) \bigcap \{\prod_{j=1}^m x_j \ge 1\}$. $\square$

We do not know of any cut separation results regarding multi-branch disjunctive cuts and nonlinear split disjunctive cuts. However, since it was proven in [6,7] that the cut separation problem of 1-branch split cuts is NP-complete, we believe that the cut separation problems associated with the multi-branch disjunctive cuts and nonlinear split disjunctive cuts are also hard in general.

The question of deriving cuts stronger than those generated from split disjunctions is an area of intense research. For Cook, Kannan and Schrijver's example, the question reduces to that of deriving $y \le 0$. We mention next some results along this line.

Under the assumption of bounded integer variables, Owen and Mehrotra [8] developed a convexification procedure for the mixed-integer programs that is based on sequential application of the variable disjunctions in the form of $(x_j \le \beta) \vee (x_j \ge \beta + 1)$, where $\beta$ is an integer. The procedure converges in the limit to the integral hull, so they call this procedure an $\infty$-convergent convexification procedure.

Andersen, Louveaux, Weismantel and Wolsey [2] explored the set of mixed-integer feasible solutions for two rows of a simplex tableau, and gave a geometrical characterization of its facets. They also show that the inequality $y \le 0$ for the Cook, Kannan and Schrijver's example can be easily derived using their results. Further, they relate the derivation of these inequalities to lattice-free bodies, which generalize split disjunctions.

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**Table 1**
Multipliers in the proof of Lemma 5

| $\bar{c}_1$ | $\bar{c}_2$ | $\bar{c}_3$ | $\bar{c}_4$ | $\psi_1$ | $\psi_2$ | $\psi_3$ | $\psi_1+\psi_2+\psi_3$ (absolute error $< 10^{-3}$) |
|---|---|---|---|---|---|---|---|
| $\bar{a}_1$ | $\bar{a}_2$ | $\bar{a}_3$ | $\bar{a}_4$ | $\frac{30}{455}$ | $\frac{64}{455}$ | $\frac{64}{455}$ | 0.347 |
| $\bar{b}_1$ | $\bar{a}_2$ | $\bar{a}_3$ | $\bar{a}_4$ | $\frac{24}{365}$ | $\frac{52}{365}$ | $\frac{52}{365}$ | 0.351 |
| $\bar{a}_1$ | $\bar{b}_2$ | $\bar{a}_3$ | $\bar{a}_4$ | $\frac{210}{3425}$ | $\frac{480}{3425}$ | $\frac{512}{3425}$ | 0.351 |
| $\bar{b}_1$ | $\bar{b}_2$ | $\bar{a}_3$ | $\bar{a}_4$ | $\frac{84}{1375}$ | $\frac{195}{1375}$ | $\frac{208}{1375}$ | 0.354 |
| $\bar{a}_1$ | $\bar{a}_2$ | $\bar{b}_3$ | $\bar{a}_4$ | $\frac{210}{3425}$ | $\frac{512}{3425}$ | $\frac{480}{3425}$ | 0.351 |
| $\bar{b}_1$ | $\bar{a}_2$ | $\bar{b}_3$ | $\bar{a}_4$ | $\frac{84}{1375}$ | $\frac{208}{1375}$ | $\frac{195}{1375}$ | 0.354 |
| $\bar{a}_1$ | $\bar{b}_2$ | $\bar{b}_3$ | $\bar{a}_4$ | $\frac{6}{107}$ | $\frac{16}{107}$ | $\frac{16}{107}$ | 0.355 |
| $\bar{b}_1$ | $\bar{b}_2$ | $\bar{b}_3$ | $\bar{a}_4$ | $\frac{24}{430}$ | $\frac{65}{430}$ | $\frac{65}{430}$ | 0.358 |
| $\bar{a}_1$ | $\bar{a}_2$ | $\bar{a}_3$ | $\bar{b}_4$ | $\frac{5}{26}$ | $\frac{2}{26}$ | $\frac{2}{26}$ | 0.346 |
| $\bar{b}_1$ | $\bar{a}_2$ | $\bar{a}_3$ | $\bar{b}_4$ | $\frac{14}{73}$ | $\frac{6}{73}$ | $\frac{6}{73}$ | 0.356 |
| $\bar{a}_1$ | $\bar{b}_2$ | $\bar{a}_3$ | $\bar{b}_4$ | $\frac{260}{1370}$ | $\frac{105}{1370}$ | $\frac{112}{1370}$ | 0.348 |
| $\bar{b}_1$ | $\bar{b}_2$ | $\bar{a}_3$ | $\bar{b}_4$ | $\frac{104}{550}$ | $\frac{45}{550}$ | $\frac{48}{550}$ | 0.358 |
| $\bar{a}_1$ | $\bar{a}_2$ | $\bar{b}_3$ | $\bar{b}_4$ | $\frac{260}{1370}$ | $\frac{112}{1370}$ | $\frac{105}{1370}$ | 0.348 |
| $\bar{b}_1$ | $\bar{a}_2$ | $\bar{b}_3$ | $\bar{b}_4$ | $\frac{104}{550}$ | $\frac{48}{550}$ | $\frac{45}{550}$ | 0.358 |
| $\bar{a}_1$ | $\bar{b}_2$ | $\bar{b}_3$ | $\bar{b}_4$ | $\frac{80}{428}$ | $\frac{35}{428}$ | $\frac{35}{428}$ | 0.350 |
| $\bar{b}_1$ | $\bar{b}_2$ | $\bar{b}_3$ | $\bar{b}_4$ | $\frac{32}{172}$ | $\frac{15}{172}$ | $\frac{15}{172}$ | 0.360 |

## Appendix

See Table 1.

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